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SOPHUS LIE'S TRANFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

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I.

1. Without entering unnecessarily into definitions which will occur more properly later, the following paragraph may serve for purposes of orientation. Among the most important notions of modern pure mathematics are the idea of a *group* and its associated notions *transformation*, *substitution*, *invariant* and *differential invariant*. Groups fall naturally and historically into two classes, *discontinuous* and *continuous*. The former are usually called *substitution groups* and are not infrequently referred to as GALOIS' groups ; the latter are known as *continuous transformation groups* and may with propriety be called LIE groups. Substitution groups find their greatest usefulness in the theory of algebraic equations. with a limited range of application to geometry ; transformation groups play a similar rôle in the theory of differential equations, with a wide application to geometry and mechanics. The idea of a substitution group in its modern signification and in its relation to the theory of algebraic equations is due to GALOIS ; LIE, after having modified and extended the idea of a substitution group, introduced the new notion into the domain of analysis and geometry and thus created his theory of transformation groups.

The great fruitfulness and remarkable simplicity of LIE's theories are their most striking characteristics. Because of their manifold applications, a thorough and systematic study of the fundamental properties of continuous groups is certain to yield the reader a liberal education in mathematics ; in addition to a knowledge of the technicalities of the theory of groups, there is gained at the same time a properly proportioned perspective of the many fields of the science brought into one domain. The group idea is a unifying principle which tends to reduce the various and in many cases apparently heterogeneous subjects of mathematics into a homogeneous body of doctrine.

It is the purpose of these notes to present some of the more elementary theorems of LIE's theories and to call attention to a few of the many applications to geometry and differential equations. The material has been drawn from the numerous published treatises* and memoirs of LIE and from his lectures delivered at the University of Leipsic in 1895 and '96. In order to an intelligent perusal of the sequel no more is required than a familiarity with the facts and processes of elementary mathematics including the simpler operations of the differential and integral calculus.

2. The simplest LIE groups are those of one parameter ; however, before proceeding to the fundamental theorems of the theory of groups of one parameter a few examples already familiar to the reader of analytical geometry as transformations of coördinates will be useful in introducing the notions.

For the sake of simplicity let the study be made in the plane. Consider the plane as a manifoldness of points, *i. e.* as a space whose space element is a point. Since it takes two independent coördinates to fix the position of a point in the plane we may say that there are ∞^2 points† in the plane or, what amounts to the same thing, that the plane is a two-dimensional space if the point is its space element. Consider the ensemble of all points of the plane and suppose that this aggregate be moved a given distance in a given direction. By this *translation* every point in the plane will be carried into the position of one of the others. In order to represent this analytically, let us suppose that the x-axis of a Cartesian coördinate system lies in the direction of the translation and that the distance through which all the points of the plane are moved is a, then the point (x, y) is carried over into the point

$$x_1 = x + a, \quad y_1 = y.$$

The segment a can be given all values from $-\infty$ to $+\infty$, and if a be varied in this manner we obtain ∞^1 translations in one and the same direction or in its opposite direction.

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*A list of these treatises is to be found in the June (1897) number of the Bulletin of the American Mathematical Society or in Teubner's catalogue. The reader who desires to prosecute the study of the subject further than the scope of these notes will find the following order of attack on Lie's published works the most satisfactory: 1° Lectures on Differential Equations with Known Infinitesimal Transformations; 2° Lectures on Continuous Groups; 3° Geometry of Contact Transformations; 4° Theory of Transformation Groups, the three volumes of this treatise in their order.

†This notation is very convenient. Its general form is—If a configuration depends on n independent parameters, of which none is superfluous, the configuration assumes ∞^n positions if the parameters are allowed to vary from $-\infty$ to $+\infty$. So, for example, there are ∞^1 points on a line, ∞^2 in the plane, ∞^3 in space, since the position of the point depends on one, two, or three parameters, respectively. Similarly there are ∞^3 circles in the plane, ∞^4 straight lines in space, ∞^5 conics in the plane, and so on. The symbol ∞^n in this connection is read "n-ply infinite number of."

Suppose now that two of these translations be carried out in succession, the first through the distance a changes the point (x, y) into the point

$$x_1 = x + a, \quad y_1 = y,$$

and the second through the distance a_1 carries the new point (x_1, y_1) into the position

$$x_2 = x_1 + a_1, \quad y_2 = y_1,$$

which together with (x, y) and (x_1, y_1) lies on a parallel to the x-axis. Now it is clear geometrically, that the passage from the initial position (x, y) to the final position (x_2, y_2) can be effected by a single translation through the distance $a + a_1$, and in fact simultaneously for all points of the plane. This also appears analytically from the fact that the elimination of the intermediate position (x_1, y_1) from the above equation gives

$$x_2 = x + a + a_1, \quad y_2 = y.$$

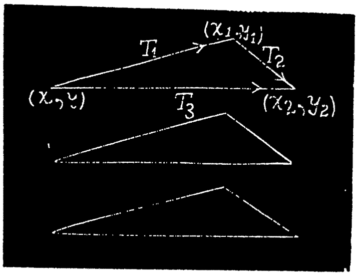
This very simple result may be formulated in the following manner :

The successive application of two translations of the family of ∞^1 *translations*

$$x_1 = x + a, \quad y_1 = y$$

is equivalent to a translation belonging to the same family.

For this reason the family is called a *group of translations.* It contains one arbitrary parameter a and hence ∞^1 translations ; accordingly it is said to be a *one-parameter group.*



3. So far the translations have been limited in direction ; let us now consider all translations in the plane. As in the preceding case let all the points of the plane be moved through the same distance a and in the same direction α ; if a and α be given all possible values we obtain a family of ∞^2 translations which includes the preceding family as a particular case. Any one of the translations of the family changes the point (x, y) into the point

$$x_1 = x + a, \quad y_1 = y + b,$$

where a and b are two arbitrary values but remain the same for all points of the plane. If a translation, T_1, carry the point (x, y) into the position of the point (x_1, y_1), and a second translation, T_2, carry the point (x_1, y_1) over to (x_2, y_2), it is clear geometrically that the point (x, y) could have been carried directly to the position (x_2, y_2) by a single translation, T_3. The length and direction of this third translation T_3, equivalent to the successive application of T_1 and T_2, is found by constructing the third side of the triangle formed by the translations T_1 and T_2, or in kinematical parlance, by taking the geometric sum of T_1 and T_2. This result appears analytically by eliminating x_1, y_1 from the equations representing the translations

$$T_1, \quad x_1=x+a,\ y_1=y+b\,;$$

$$T_2, \quad x_2=x_1+a_1,\ y_2=y_1+b_1\,;$$

this elimination yields the equation

$$T_3, \quad x_2=x+a+a_1,\ y_2=y+b+b_1,$$

which is of the same form as the equations representing T_1 and T_2 and hence belongs to the same family as T_1 and T_2; therefore we conclude that

The successive application of any two translations of the family of all translations of the plane

$$x_1=x+a,\ y_1=y+b$$

is equivalent to a single translation belonging to the same family.

Because of the possession of this remarkable property* the family of all translations of the plane is called a *group of translations.* The *group* contains two arbitrary constants a and b, *i. e.* it has ∞^2 different translations; for this reason the group of all translations in the plane is called a *two-parameter* group.

4. In order to present simple concrete examples illustrative of several other fundamental notions let us return to the family of all translations parallel to the x-axis

$$x_1=x+a,\ y_1=y\,; \tag{1}$$

among these ∞^1 translations there is one to be noted, namely that one for which

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*It is easy to see that this property of the equivalence of the successive applications of any two transformations of a family of transformations to a third transformation belonging to the same family is a remarkable one, peculiar to certain families, and not common to all. For example, the equations,

$$x_1=a-x,\quad y_1=y,$$

represent a family of ∞^1 transformations, which may be readily constructed geometrically, but a transformation S_1 changing (x, y) into

$$x_1=a-x,\quad y_1=y,$$

followed by S_2, changing (x_1, y_1) into

$$x_2=a_1-x_1,\quad y_2=y,$$

produces, by the elimination of (x_1, y_1) from these equations, the equations

$$x_2=(a_1-a)+x,\quad y_2=y,$$

which represents the transformation S_3 equivalent to the successive application of S_1 and S_2. But the x of the original family is equal to a constant minus the old x, while in S_3 the new x is equal to a constant plus the old x, hence S_3 does not belong to the same family as S_1 and S_2. The ∞^1 transformations represented by the above equation then do not form a LIE group.

$a=0$, *i. e.* a translation through the distance zero. By this translation all points of the plane remain at rest, and strictly speaking the term translation is no longer allowable. If, for the sake of continuity, the term translation is to be retained as applicable to this case also, then the translation by which every point is changed into itself is called the *identical translation.* It is to be further remarked that for every translation of this group there is a translation of the group which, carried out after the former, cancels its effect. Thus the successive application of the translations corresponding to $+a$ and to $-a$ respectively is equivalent to the translation $a-a=0$, that is, to the identical transformation. For this reason the two translations are said to be inverse.

If we put a equal to an infinitely small constant ∂t, we obtain an *infinitesimal translation*, which gives to all points of the plane only an infinitely small motion

$$x_1=x+\partial t, \quad y_1=y.$$

By this translation the coördinates x, y receive infinitely small increments

$$\partial x=\partial t, \quad \partial y=0,$$

and if the *infinitesimal translation* be carried out n times successively, the point (x, y) is changed into

$$x_1=x+n\partial t, \quad y_1=y\ ;$$

if the infinitesimal translation be repeated an infinite number of times, or, what comes to the same thing, if n becomes infinite, then $n\partial t$ is equal to some finite quantity a and we have again a finite translation

$$x_1=x+a, \quad y_1=y.$$

We shall find later on that a one-parameter group contains but one infinitesimal transformation.

Suppose that we operate on a definite point (x_0, y_0) with all translations of the one-parameter group (1) ; the point will take ∞^1 different positions :

$$x=x_0+a, \quad y=y_0,$$

the aggregate of which is a parallel to the x-axis. This line, the locus of all the points into which a definite point is changed by operating on it with all the translations of the group, is called the *path curve of the point*, or *path curve of the one-parameter group*. There are altogether ∞^1 path curves of the group (1) consisting of the family of straight lines parallel to the x-axis.

Any translation of the group carries any one of the path curves, as a whole, forward in its own direction a distance a ; *i. e.* the path curve as a whole remains at rest. The path curves are invariant by all the translations of the group. In addition to the line at infinity and the path curves, there is no other *invariant curve* by this one-parameter group, *i. e.* no other curve all of whose points are changed into points of the same curve by all the transformations of the group.

If a function of (x, y), $F(x, y)$ is to be invariant by the group

$$x_1 = x + a, \quad y_1 = y,$$

we must have $F(x_1, y_1) \equiv F(x+a, y) = F(x, y)$, for all values of a. In order to determine the function F we need only to take an infinitesimal value for a, and carry out the infinitesimal translation $x_1 = x + \partial t$, $y_1 = y$.* Taylor's series gives

$$F(x, y) + \frac{a}{1}\frac{\partial F(x, y)}{\partial x} + \frac{a^2}{1.2}\frac{\partial^2 F(x, y)}{\partial x^2} + \ldots\ldots = F(x, y),$$

or cancelling $F(x, y)$ from each side and neglecting terms of the second order

$$\frac{\partial F(x, y)}{\partial x} = 0,$$

that is, F does not contain x and is a function of y alone. Hence every function $F(y)$ is an *invariant function* by the one-parameter group (1). An *invariant function* equated to a constant gives an *invariant equation*, which represents one or more *path curves* of the group.

The reader may find it interesting to verify the group property for the following families and to determine the path curves and forms of invariant functions :

1° Rotations about a fixed point $\begin{cases} x_1 = x\cos\alpha - y\sin\alpha, \\ y_1 = x\sin\alpha + y\cos\alpha\,; \end{cases}$
2° The affine transformations $x_1 = ma, \; y_1 = y\,;$
3° The perspective transformations $x_1 = ax, \; y_1 = ay\,;$
4° The transformations $x_1 = ax, \; y_1 = y/a\,;$
5° The group of all Euclidean motions in ordinary space

$$x_1 = a_1 x + a_2 y + a_3 z + a_0,$$

$$y_1 = b_1 x + b_2 y + b_3 z + b_0,$$

$$z_1 = c_1 x + c_2 y + c_3 z + c_0.$$

The University of Chicago, 10 September, 1897.

[To be Continued.]

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*In order that a function, equation or curve be invariant by all of the finite transformations of a one-parameter group, it is necessary and sufficient that the function, equation or curve be invariant by the infinitesimal transformation of the group. This theorem will be proved in the sequel.